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An Operator Inequality and Matrix Normality

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ABSTRACT

Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$; denote $|A| = (A^*A)^{1/2}$ and the norm of $x \in \mathcal{H}$ by $\|x\|$. It is proved that

$$|(Au, v)| \le \||A|^{\alpha}u\| \, \||A^*|^{1-\alpha}v\| \qquad \forall u, v \in \mathcal{H}$$

for any $0 < \alpha < 1$. In particular,

$$|(Au, v)| \le (|A|u, u)^{1/2}(|A^*|v, v)^{1/2} \qquad \forall u, v \in \mathcal{H}.$$

When $\mathcal{H}$ is of finite dimension, it is shown that A must be a normal operator if it satisfies

$$|(Au, u)| \le (|A|u, u)^{\alpha}(|A^*|u, u)^{1-\alpha} \qquad \forall u \in \mathcal{H},$$

for some real number $\alpha \neq \frac{1}{2}$.

1. NOTATION

Let $\mathcal{H}$ be a Hilbert space over the complex number field C, and $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$. In case $\mathcal{H}$ is of

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finite dimension n, we identify $\mathcal{B}(\mathcal{H})$ with $M_n(C)$, the collection of all $n \times n$ complex matrices. C^n denotes the set of all complex column vectors with n components. For $A \in \mathcal{B}(\mathcal{H})$, let $|A| = (A^*A)^{1/2}$, where A^* is the adjoint of A. It is immediate that $|UAV| = V^*|A|V$ for any U and V unitary, and that A is normal if and only if $|A| = |A^*|$.

Throughout this paper $A \geq 0$ means that A is a positive operator, i.e., $(Au, u) \geq 0$ for all $u \in \mathcal{H}$, and $\|x\|$ denotes the norm of $x \in \mathcal{H}$.

2. AN OPERATOR INEQUALITY

We first give an operator inequality.

THEOREM 1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then for any $\alpha \in (0,1)$*

$$|(Au, v)| \leq \| |A|^\alpha u \| \| |A^*|^{1-\alpha} v \| \qquad \forall u, v \in \mathcal{H}. \tag{1}$$

In particular,

$$|(Au, v)| \leq (|A|u, u)^{1/2} (|A^*|v, v)^{1/2} \qquad \forall u, v \in \mathcal{H}. \tag{2}$$

Proof. Let $A = U|A|$ be the polar decomposition of A (see [1, p. 248] or [5, p. 197]), where U is a partial isometry such that U^*U is a projection onto $(\ker A)^\perp$. Noticing that

$$\ker A = \ker |A| = \ker |A|^{1/2} = \cdots = \ker |A|^{1/2^m}$$

for any positive integer m, we have

$$U^*U|A|^\alpha = U^*U|A|^{1/2^m}|A|^{\alpha - 1/2^m} = |A|^\alpha \qquad \forall \alpha \in (0,1),$$

when m is so large that $1/2^m < \alpha$, and

$$A = U|A|^{1-\alpha}U^*U|A|^\alpha = |A^*|^{1-\alpha}U|A|^\alpha.$$

By the Cauchy-Schwarz inequality we have

$$|(Au, v)| = (U|A|^\alpha u, |A^*|^{1-\alpha} v) \leq \| |A|^\alpha u \| \, \| |A^*|^{1-\alpha} v \|.$$

Taking $\alpha = \frac{1}{2}$ gives (2). ∎

It is immediate from (2) that

$$|(Au, u)| \leq (|A|u, u) \qquad \forall u \in \mathcal{H}, \tag{3}$$

provided that A is normal.

3. MATRIX NORMALITY

The shift operator $Ae_i = e_{i+1}$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis in a separable Hilbert space $\mathcal{H}$, illustrates that (3) holds for some nonnormal operators. This happens only in infinite-dimensional spaces, as we will show in the following theorem that $\frac{1}{2}$ is the only possible value such that (2) holds for nonnormal (finite) matrices. Thus one more necessary and sufficient condition for a matrix to be normal is added to the list in [2].

THEOREM 2. *Let $A \in M_n(C)$. If α is a real number different from $\frac{1}{2}$, and if*

$$|(Au,u)| \leq (|A|u,u)^{\alpha}(|A^*|u,u)^{1-\alpha} \qquad \forall u \in C^n, \tag{4}$$

then A is normal.

We break down the proof of Theorem 2 into two cases: A is nonsingular and A is singular.

For the case where A is nonsingular, we need a lemma.

LEMMA 1. *If $A = (a_{ij})$ is an n-square complex matrix with all eigenvalues equal to 1 in absolute value, and if*

$$|(Ay,y)| \leq (Ay,Ay)^{\alpha} \qquad \text{for all unit vectors } y \in C^n, \tag{5}$$

then either $\alpha = \frac{1}{2}$ or A is a unitary matrix.

Proof of the Lemma. We may assume that A is an upper-triangular matrix with diagonal entries $1, \lambda_2, \ldots, \lambda_n$. We show that the first row of A is $(1,0,\ldots,0)$ if $\alpha \neq \frac{1}{2}$. Suppose $a_{1s} \neq 0$ for some $s > 1$. We first deal with the case $a_{1s} > 0$.

Let $y = (\cos t, 0, \ldots, 0, \sin t, 0, \ldots, 0)^T$, where $\sin t$ is placed in the sth position, and define, for each α,

$$\begin{aligned} f_\alpha(t) &= (Ay,Ay)^{\alpha} - |(Ay,y)| \\ &= (Ay,Ay)^{\alpha} - [(Ay,y)(\overline{A}y,y)]^{1/2}, \end{aligned}$$

where $\overline{A}$ is the conjugate of A.

Obviously $f_\alpha(0) = 0$, since $Ay = (1,0,\ldots,0)^T$ when $y = (1,0,\ldots,0)^T$. Noting that

$$\begin{aligned} \frac{df_\alpha}{dt} &= \alpha(Ay,Ay)^{\alpha-1}[(Ay',y) + (Ay,Ay')] - \tfrac{1}{2}[(Ay,y)(\overline{A}y,y)]^{-1/2} \\ &\quad \times \{[(Ay',y) + (Ay,y')](\overline{A}y,y) + [(\overline{A}y',y) + (\overline{A}y,y')](Ay,y)\}, \end{aligned}$$

where $y' = (-\sin t, 0, \ldots, 0, \cos t, 0, \ldots, 0)^T$ is the derivative of y, we easily compute that

$$\left.\frac{df_\alpha}{dt}\right|_{t=0} = (2\alpha - 1)a_{1s}.$$

Thus $df_\alpha/dt|_{t=0}$ is either positive or negative if $\alpha \neq \frac{1}{2}$, and there is an $\epsilon > 0$ such that $df_\alpha/dt > 0$ ($df_\alpha/dt < 0$, respectively) for all $t \in (-\epsilon, \epsilon)$. Hence $f_\alpha(t)$ cannot be always nonnegative, a contradiction.

If a_{1s} is a complex number rather than a positive number, one uses the same argument for P^*AP in place of A, where P is the matrix obtained from the identity by substituting 1 in the sth row with $e^{i\theta}$, where $a_{1s} = |a_{1s}|e^{i\theta}$.

We thus reduce the problem to the remaining $(n-1) \times (n-1)$ matrix, and an inductive argument gives that A is unitarily diagonalizable, i.e., A is unitary, since all its eigenvalues are equal to one in absolute value (assumption). ∎

We are now ready to prove the theorem.

Proof of Theorem 2. We first consider the case where A is nonsingular.

Let $A = UDV$ be the singular-value decomposition of A, where D is diagonal and invertible, and U and V are unitary. Note that $|A| = V^*DV$ and $|A^*| = UDU^*$. The original inequality (4) becomes

$$|(UDVu, u)| \leq (V^*DVu, u)^\alpha (UDU^*u, u)^{1-\alpha}, \tag{6}$$

or

$$|(D^{1/2}Vu, D^{1/2}U^*u)| \leq (D^{1/2}Vu, D^{1/2}Vu)^\alpha (D^{1/2}U^*u, D^{1/2}U^*u)^{1-\alpha}. \tag{7}$$

For any nonzero u, set $y = (1/\|D^{1/2}U^*u\|)D^{1/2}U^*u$. Then $\|y\| = 1$, and y ranges over all unit vectors as u runs over all nonzero vectors.

Rewriting (7) in the form

$$|(\widetilde{A}y, y)| \leq (\widetilde{A}y, \widetilde{A}y)^\alpha \qquad \text{for all unit vectors} \quad y \in C^n,$$

where $\widetilde{A} = D^{1/2}VUD^{-1/2}$, and applying Lemma 1 to $\widetilde{A}$, we get that $D^{1/2}VUD^{-1/2}$ is a unitary matrix. Thus $D^{-1/2}U^*V^*DVUD^{-1/2} = I$ and $VUD = DVU$. It is immediate that $UD^2U^* = V^*D^2V$, so $AA^* = A^*A$ and A is normal.

We next deal with the case where A is singular, using mathematical induction on n. If $n = 1$, we have nothing to prove. Suppose the conclusion is true for $(n-1)$-square matrices.

Noting that (4) still holds when A is replaced by U^*AU for any unitary matrix U, we may assume, without loss of generality, that

$$A = \begin{pmatrix} A_1 & B \\ 0 & 0 \end{pmatrix},$$

where A_1 is an $(n-1)$-square matrix and B is an column $(n-1)$-vector.

If $B = 0$, then A is normal by induction on A_1. If $B \neq 0$, we take $u_1 \in C^{n-1}$ such that $(B, u_1) \neq 0$. Let

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \text{with} \quad u_2 > 0.$$

Then

$$|(Au, u)| = |(A_1u_1, u_1) + (B, u_1)u_2|,$$

and

$$(|A^*|u, u) = ((A_1A_1^* + BB^*)^{1/2}u_1, u_1)$$

independent of u_2.

To compute $(|A|u, u)$, let

$$|A| = \begin{pmatrix} C & D \\ D^* & \beta \end{pmatrix}.$$

Then $\beta > 0$, since $B \neq 0$, and

$$(|A|u, u) = (Cu_1, u_1) + u_2[(D, u_1) + (u_1, D)] + \beta u_2^2.$$

Letting $u_2 \to \infty$ in (4) yields $2\alpha \geq 1$ or $\alpha \geq \frac{1}{2}$.

(4) can be rewritten as follows:

$$|(A^*u, u)| \leq (|A^*|u, u)^{1-\alpha}(|A|u, u)^{1-(1-\alpha)}.$$

Applying the same argument to A^* partitioned as

$$\begin{pmatrix} A_1' & B' \\ 0 & 0 \end{pmatrix},$$

one obtains $\alpha \leq \frac{1}{2}$ if $B' \neq 0$. By induction, we see that A is normal if $\alpha \neq \frac{1}{2}$. ∎

REMARK 1. A different proof for the case of $\alpha = 1$ or 0 (assume that $A \neq 0$) in Theorem 2 has been noticed. Without loss of generality, suppose that A is an upper triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Let $(t_1, \ldots, t_n)$ be the first row of $|A|$; then $|t_1|^2 + \cdots + |t_n|^2 = |\lambda_1|^2$. On the other hand, taking $u = (1, 0, \ldots, 0)^T$ gives $|\lambda_1| \leq t_1$. Thus $t_1 = |\lambda_1|$ and $t_2 = \cdots = t_n = 0$, so that $|\lambda_1|$ is a singular value of A. Then use mathematical induction.

REMARK 2. Noticing that when A is a square matrix

$$\begin{pmatrix} |A| & A^* \\ A & |A^*| \end{pmatrix} \circ \begin{pmatrix} |A^*| & A \\ A^* & |A| \end{pmatrix} \geq 0,$$

where $\circ$ stands for the Hadamard product of two matrices (see [3] or [4]), one obtains

$$|((A \circ A^*)u, u)| \leq ((|A| \circ |A^*|)u, u) \qquad \forall u \in C^n. \tag{8}$$

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